On Computing Incompressible Shear Flows with Compressible Methods

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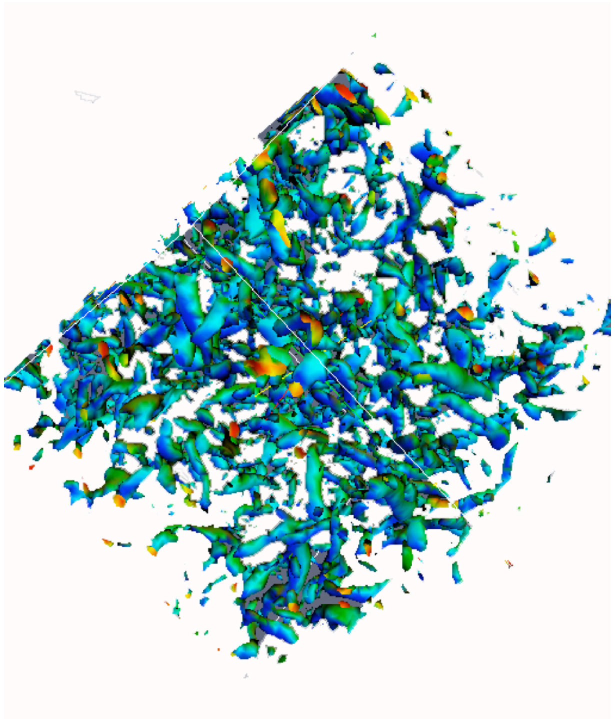
The Outline of This Talk*

- **Defining the challenge of computing incompressible shear flows with compressible codes.**
  - **Renewed focus from ILES studies of turbulence**
- **A shear test problem and its incompressible solution**
- **Problems: real and imagined**
  - **Real - an ill-conditioned system, violations of the 2nd law or lack of convergence in Mach number**
  - ***Imagined - incompressibility as an approximation and its limitations***

***The same line of investigation as Ben Thornber's talk earlier this afternoon.**

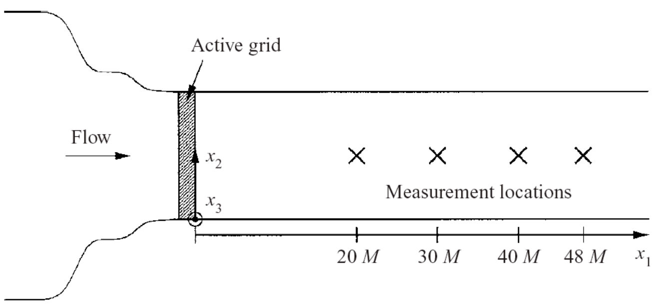
A synthesis of ILES research is found in a book with contributions from key ILES researchers.

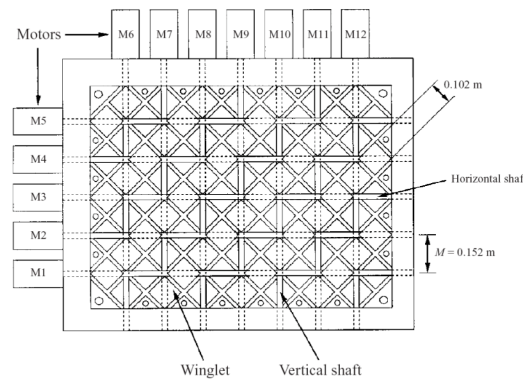


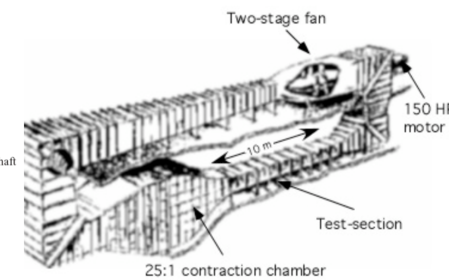
Recently there has been a renewed interest in solving incompressible flows with compressible method often in the context of turbulence.

We have had excellent results using compressible methods for turbulence.

- **Success in modeling the JHU wind tunnel experiment has been achieved.**



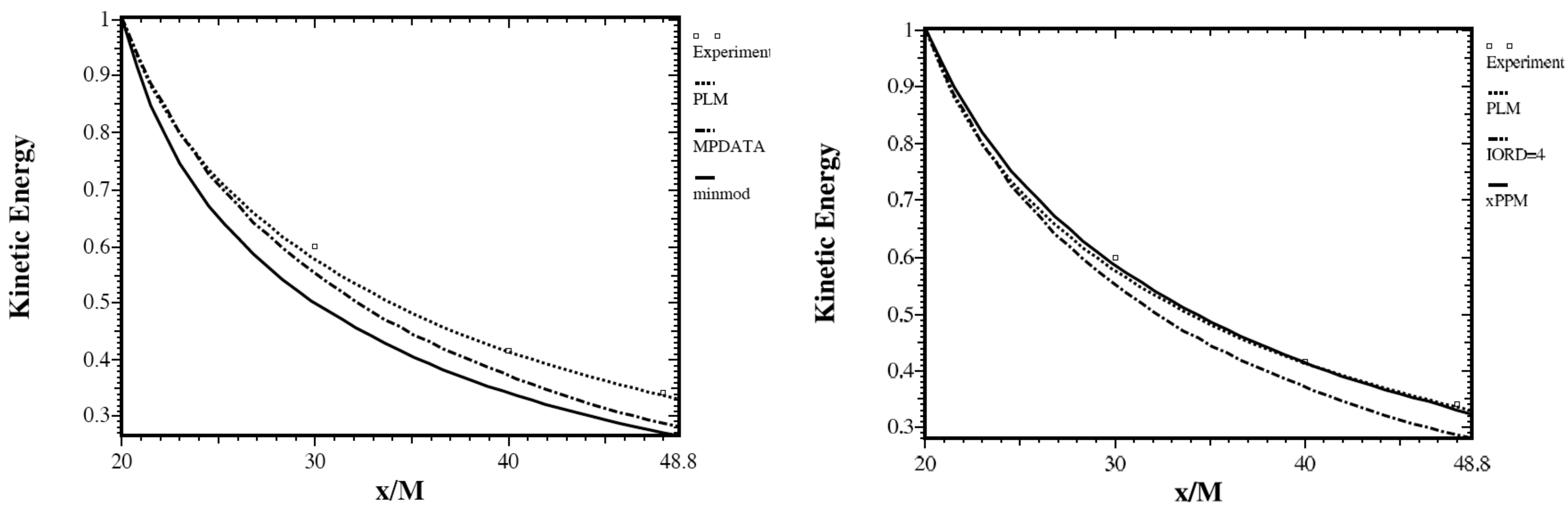




- **The experimental data was published in JFM (480, pp. 129-160, 2003) and can be found on a JHU website.**
- **A difficult decaying turbulence experiment, with lots of data for validation at Re_λ=720.**

Great results were computed considing this a low-Mach number flow (M=0.1, not incompressible)

- The kinetic energy decays were spot on with both PLM and PPM methods (not minmod PLM!).
  - The method passes an important validation hurtle



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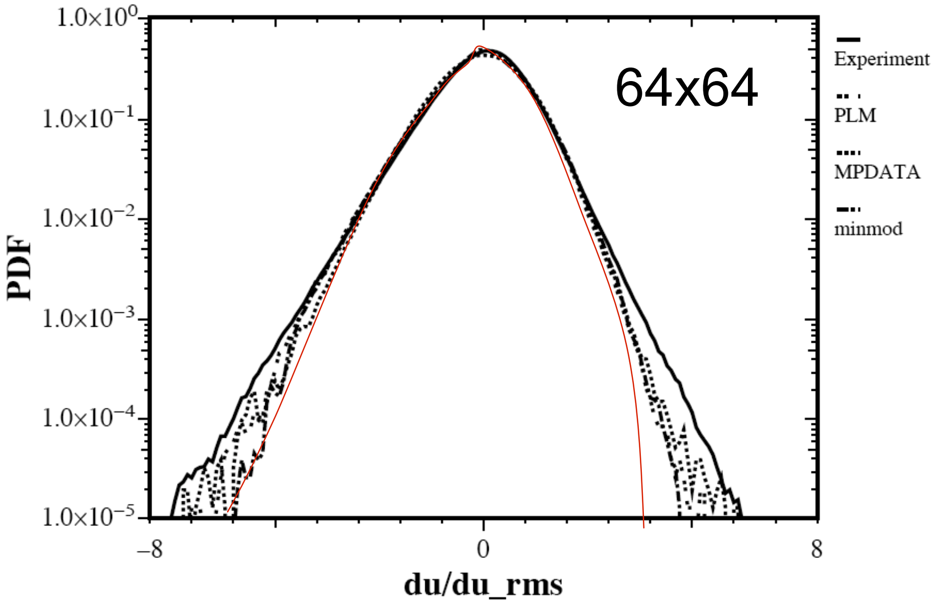
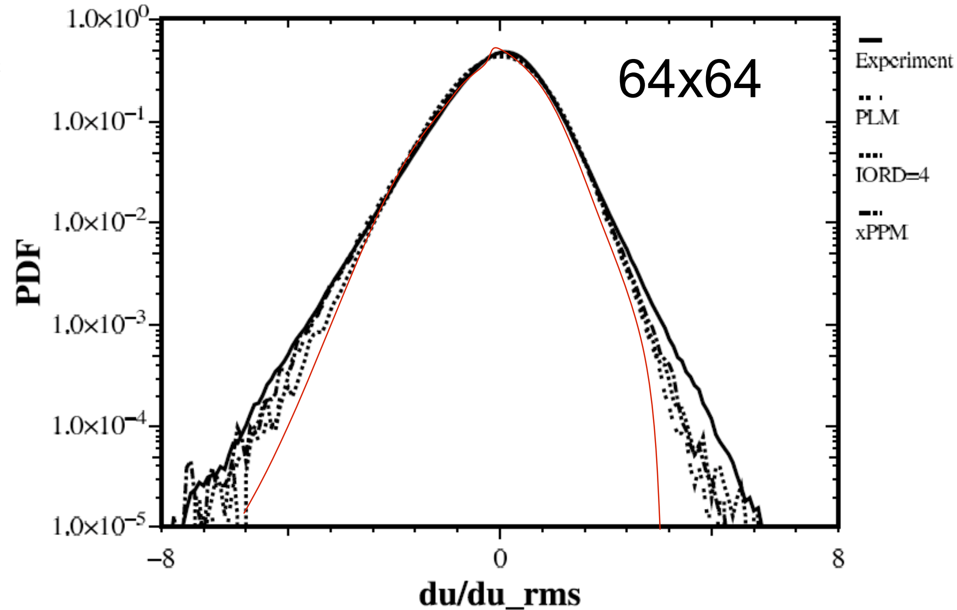
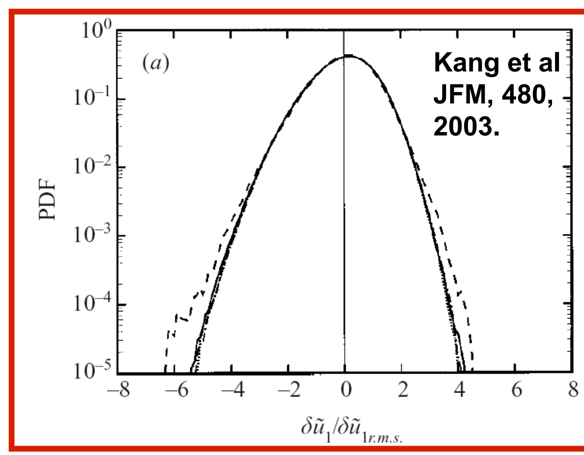
The PDF of the velocity increments are much more impressive (on a coarse grid).





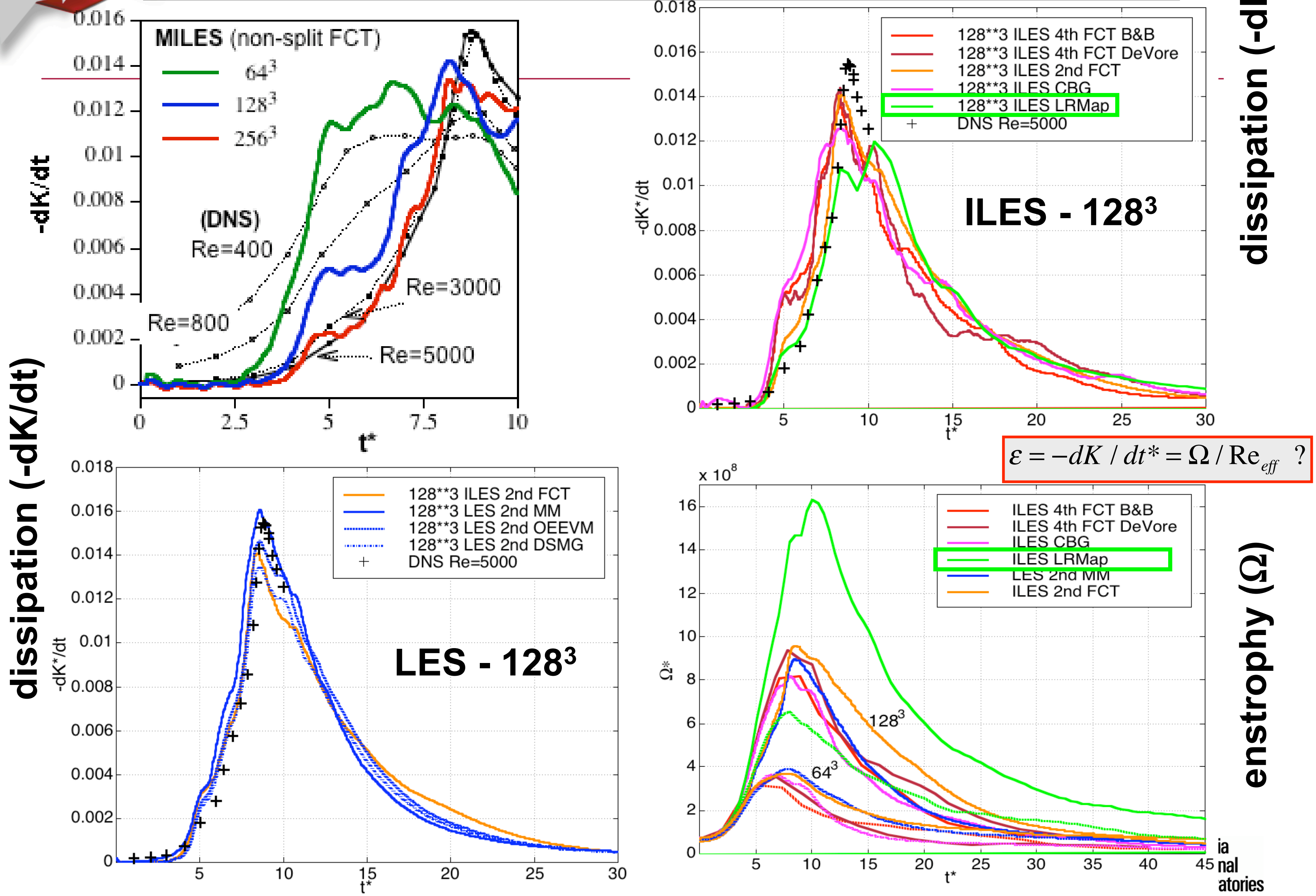
All the ILES methods produce much more intermitent results than the CLES. The xPPM and MPDATA results are the closest to the data.

128x128, LES w/dynamic-mixed



Evolution of the Taylor-Green Vortex

MILES: Grinstein, Fureby, Drikakis, Youngs, 2006 (JoT, in press); DNS, Brachet *et al.* '83, '91



$$\varepsilon = -dK / dt^* = \Omega / \mathrm{Re}_{eff} \quad ?$$

Evolution of the Taylor-Green Vortex

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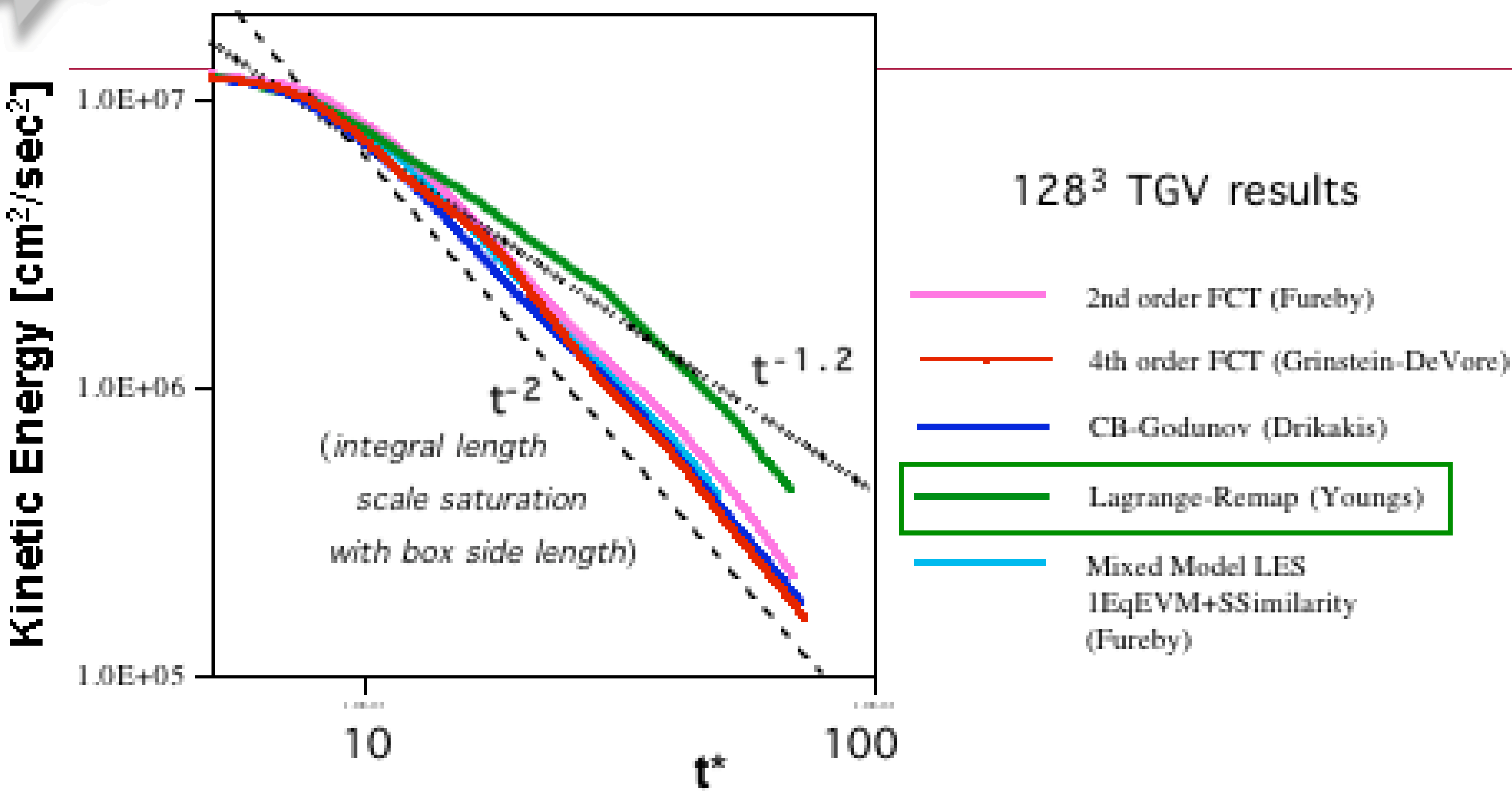


Power law decay of the mean kinetic energy

- ~ $t^{-1.2}$ behavior just after t*~9, generally accepted as characteristic of decaying turbulence
- ~ t^{-2} behavior, asymptotically, reflecting that eddies larger that box side length cannot exist
- LR is significantly less dissipative, ….

Why do Youngs' results stand out?

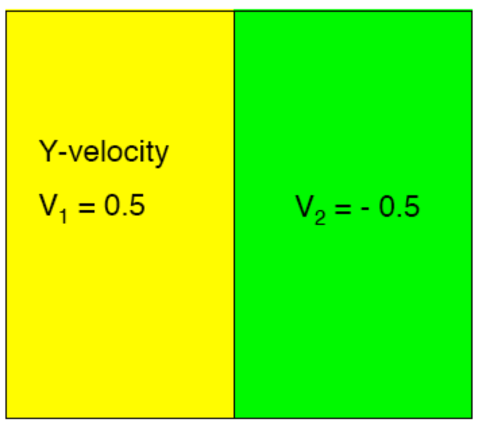
- **Is there something intrinsically "better" with Youngs' TURMOIL code?**
  - **Is it the Lagrange-Remap, 3rd order advection?**
- **Youngs has hypothesized that one reason is the form used for artificial viscosity in the Lagrangian step**
  - The artificial viscosity is proportional to the multidimensional divergence of velocity squared

$$Q = C\left(\Delta x\right)^2 \mathbf{max}\left(0, -\nabla \cdot u\right)^2$$

  - ***This form of viscosity is less favorable for shocks***
  - **Edge viscosity would not share this property, but would perform better in shocks**

Youngs has suggested abstracting this to a simpler problem, like a 2-D shear layer.

- **David Youngs of AWE developed a simple problem to examine the differences in code performance on shear.**
- **The problem is an ideal, but discontinuous shear with a potential flow perturbation in a low frequency mode**
  - **The problem exhibits different structures with varying Mach number**



$$u = \frac{\partial A_z}{\partial y} \qquad v = -\frac{\partial A_z}{\partial x}$$

$$\text{where } A_z = \frac{V_0}{k}\cos(ky)\exp(-k|x|)$$

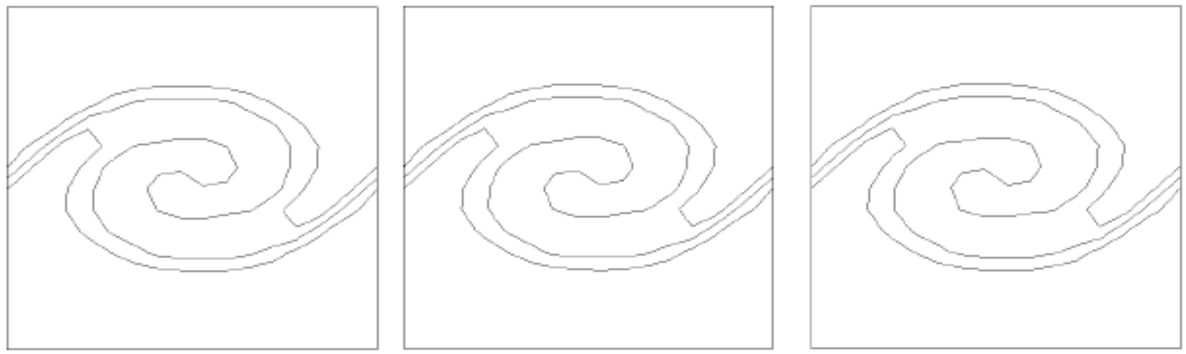
$$\text{and } V_0 = \text{amplitude of velocity perturbation}$$
$$= 0.1\ \Delta V$$

- I have decided to use a somewhat different problem.

Youngs studied a simple shear problem and found serious problems.

- **His code showed little dependence on Mach or CFL number,**

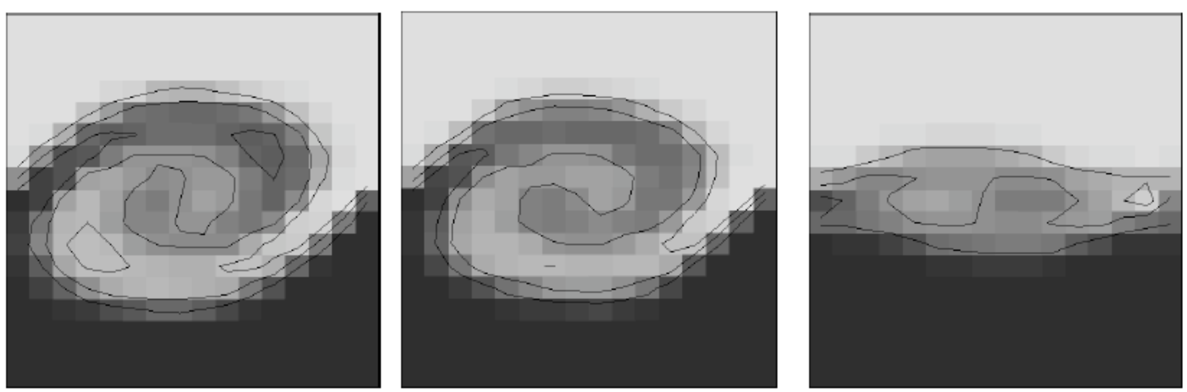
M=0.2 Δt=0.005 | M=0.2 Δt=0.0005 | M=0.02 Δt=0.0005



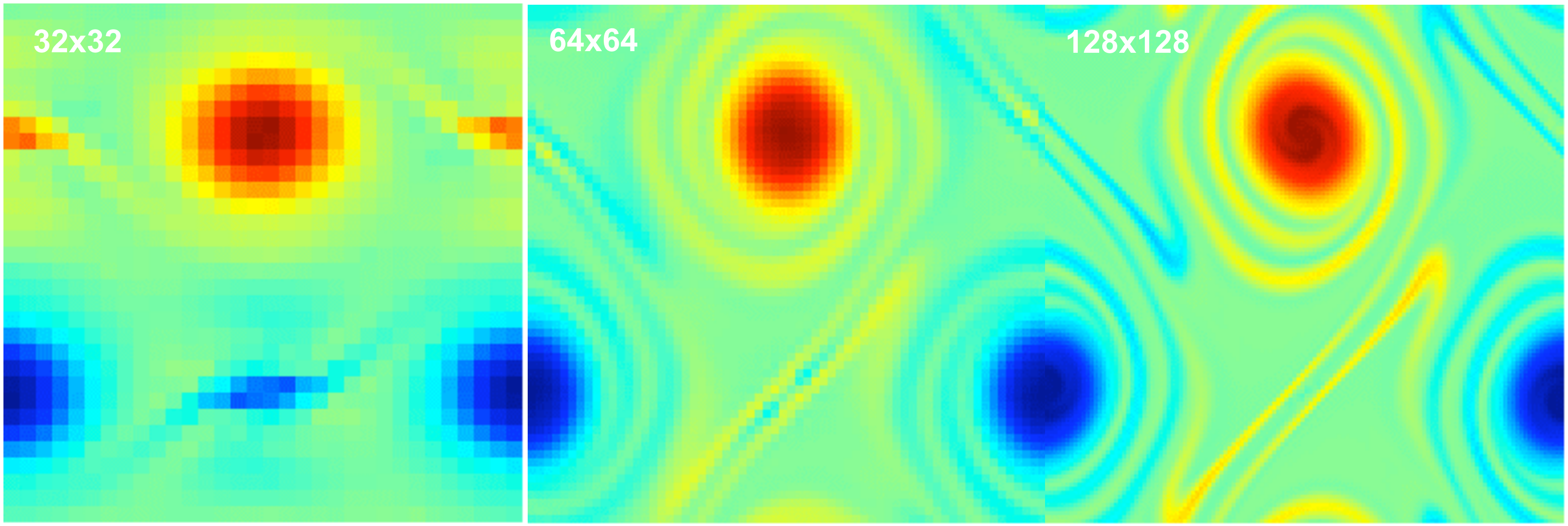
16x16 grids, Under refinement the morphology of the shear layer changes as a function of Mach number

- **But a Godunov Method (VH-1 L-R PPM) showed great dependence on both!**

M=0.2 Δt=0.005 | M=0.2 Δt=0.0005 | M=0.02 Δt=0.0005



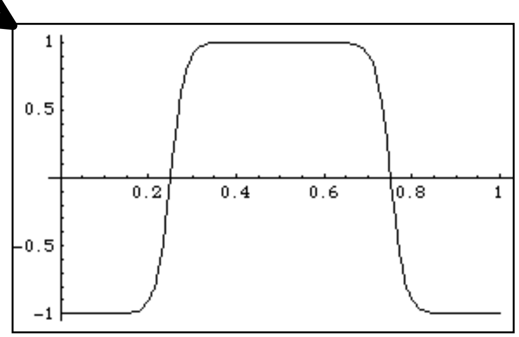
An doubly periodic shear layer as a useful test problem (usually incompressible).

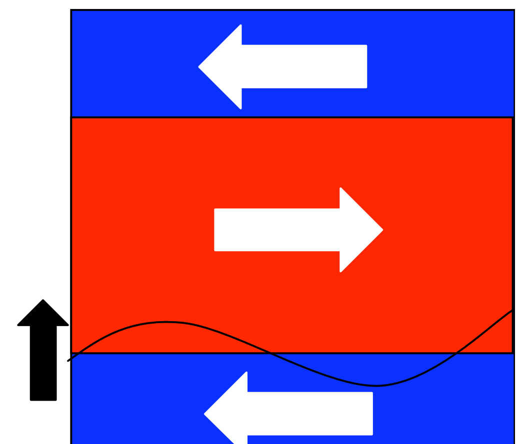


$$u = Tanh\left[30\left(\tfrac{3}{4}-y\right)\right] + Tanh\left[30\left(y-\tfrac{1}{4}\right)\right] - 1$$

$$v = 0.05\,Sin(2\pi x)$$

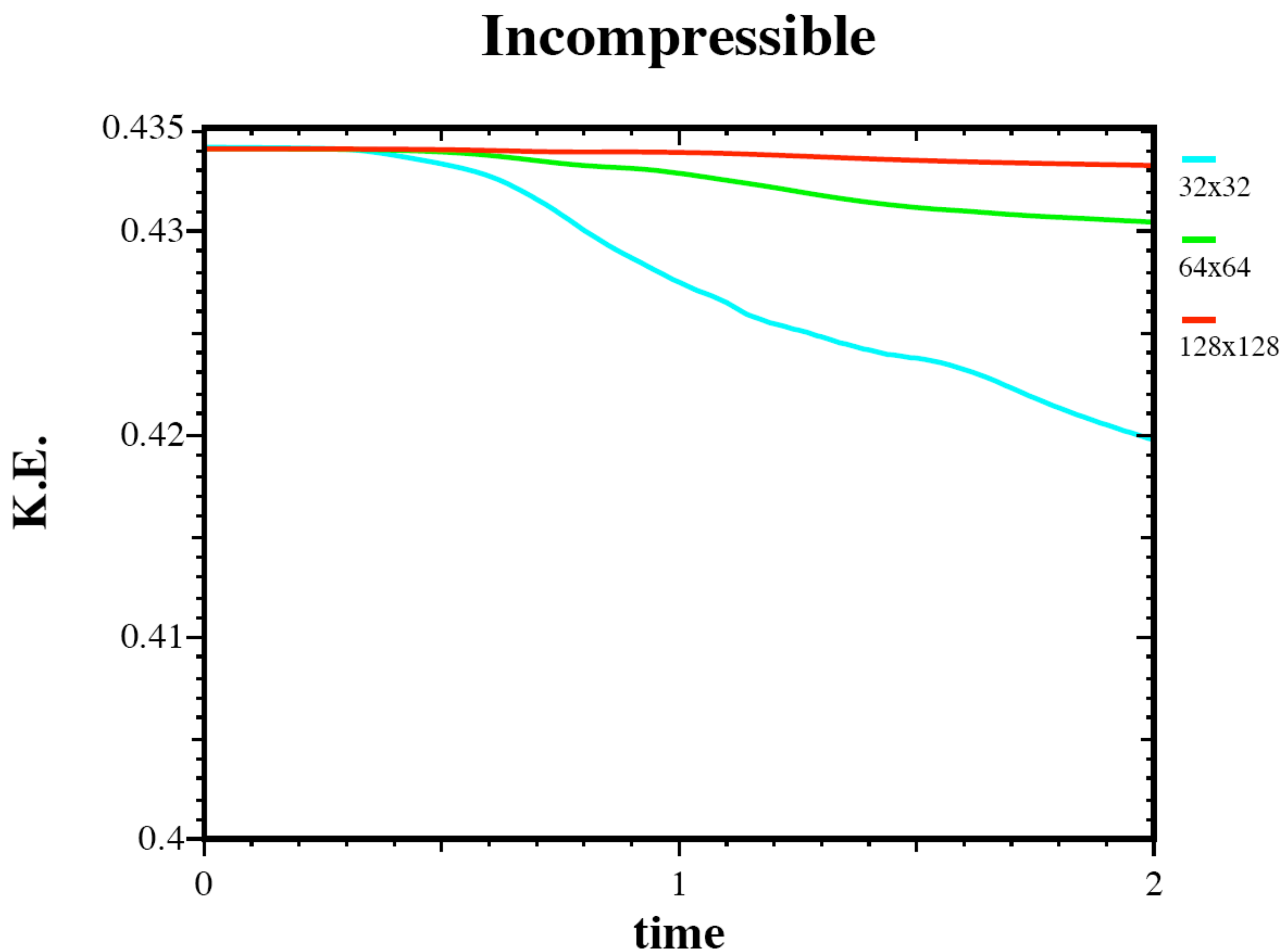
$$p = {}^{1}\!\big/_{\gamma M^{2}}$$





Kinetic energy decay converges at 2nd order under mesh refinement.

The calculation is converging toward the incompressible analytical result of kinetic energy conservation with a rate of 1.94 (2nd order).



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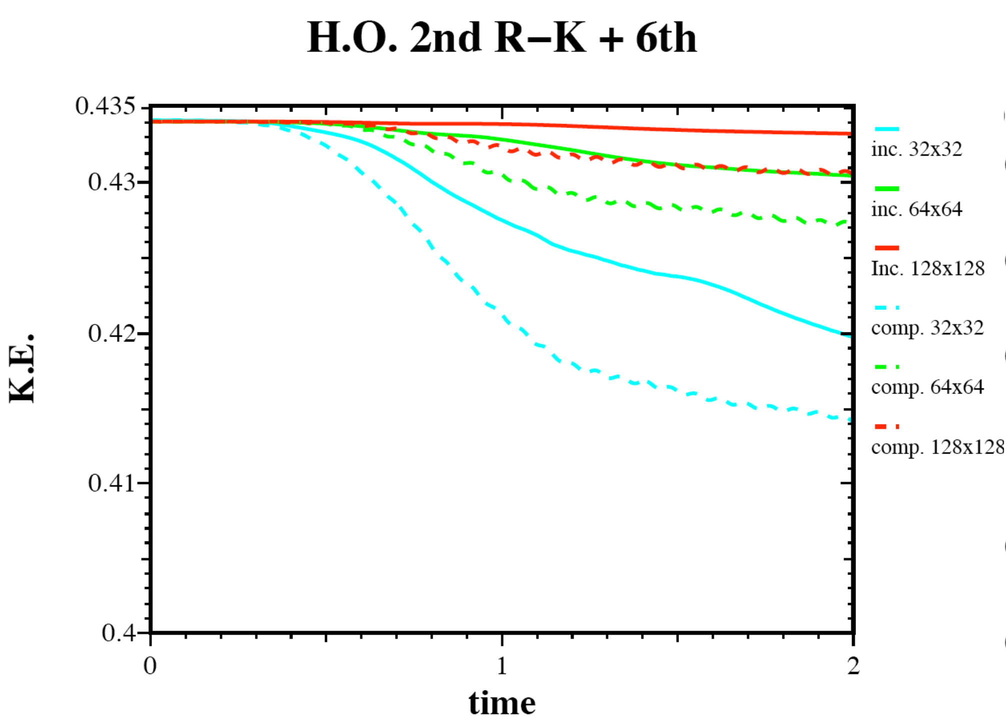
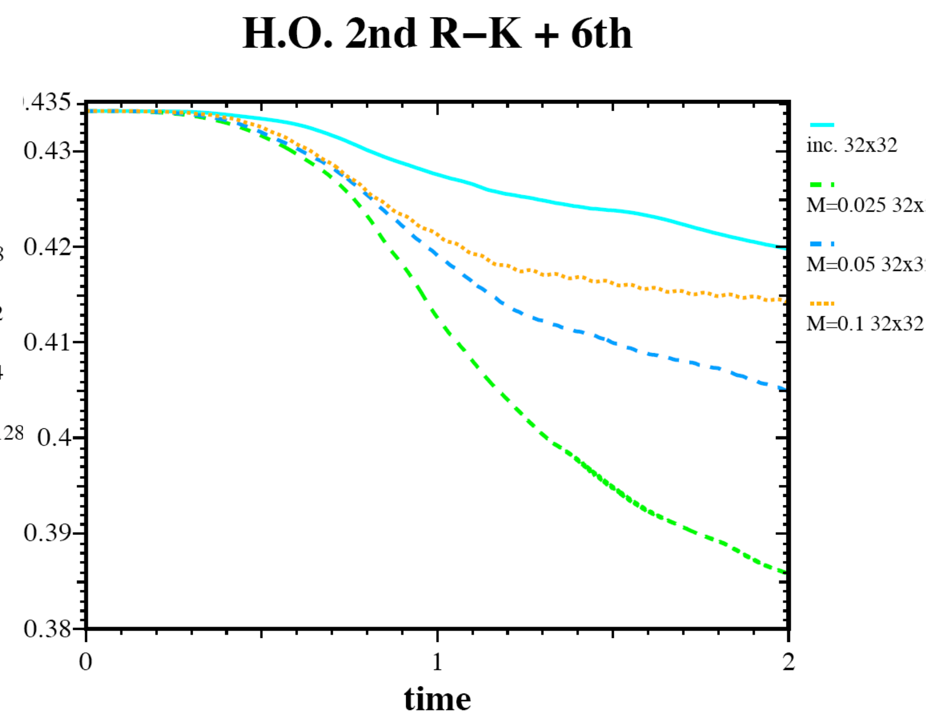
2nd Runge-Kutta plus 4th order centered edge values with monotone limiting*

- We see 2nd order convergence in KE with mesh ref.
- We see divergence as the Mach number decreases.
- Converging to a finite dissipation result.

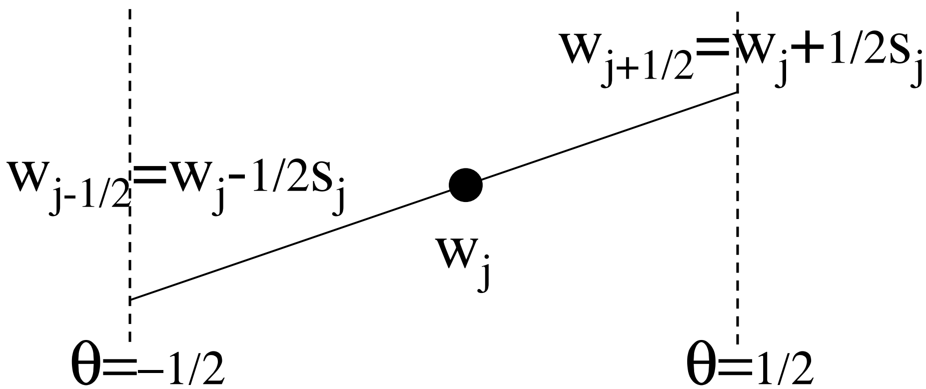




*Results that most clearly relate to Thornber's results

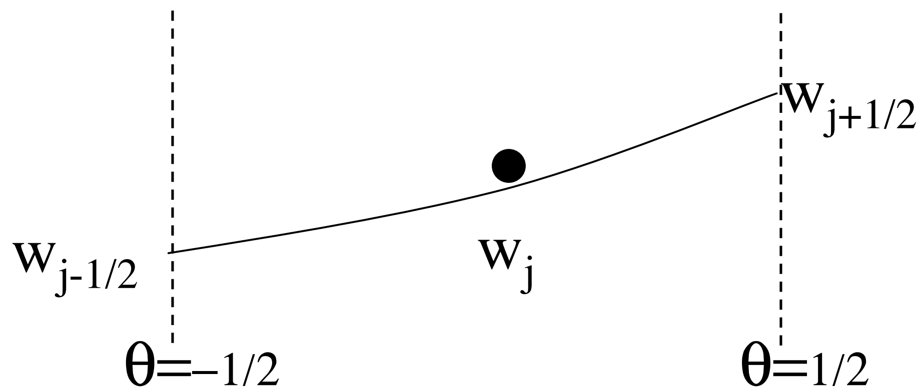
We examine compressible results with a couple of different Godunov methods.

PLM



Defined by the slope and cell-average, produces a inherently "broken space" approximation.

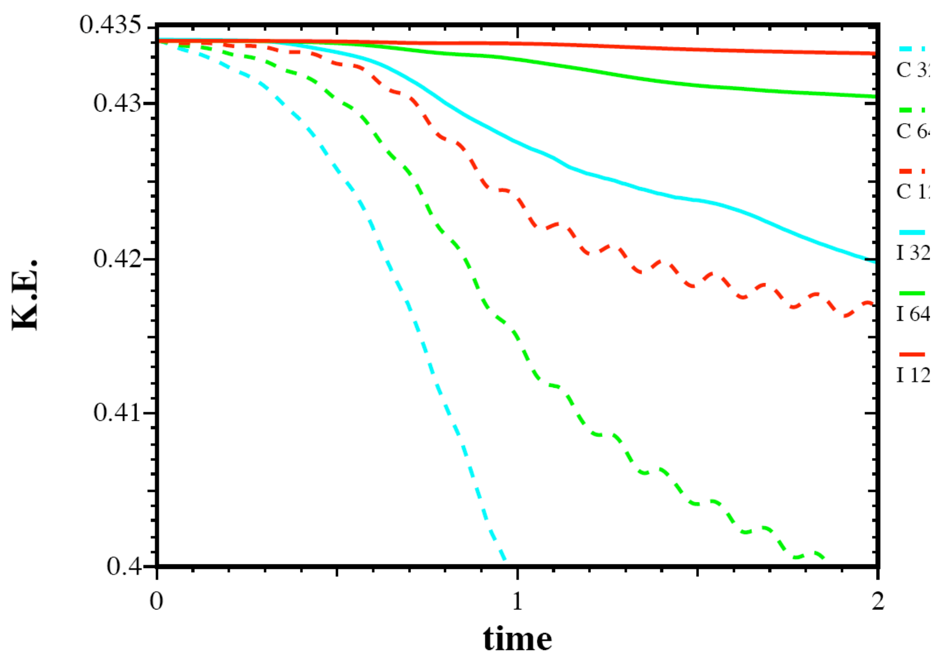
PPM



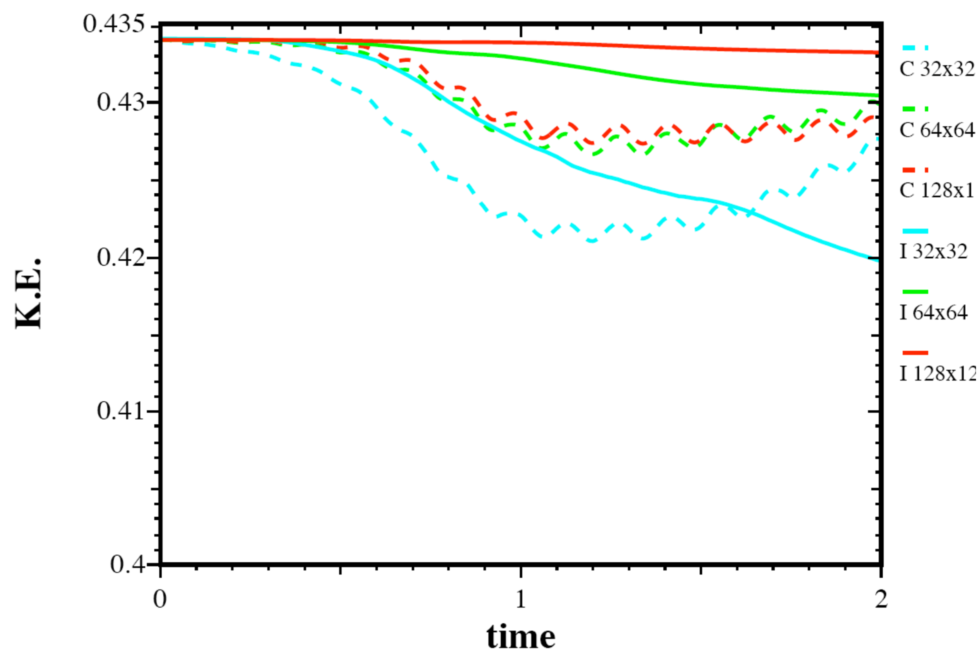
Defined by the edge values and cell-average, produces a potentially continuous approximation with centered edge values.

Performance for simple Godunov methods - PLM* w/high order or minmod.

Mesh Refinement – Minmod



Mesh Refinement – PLM 4th order slope



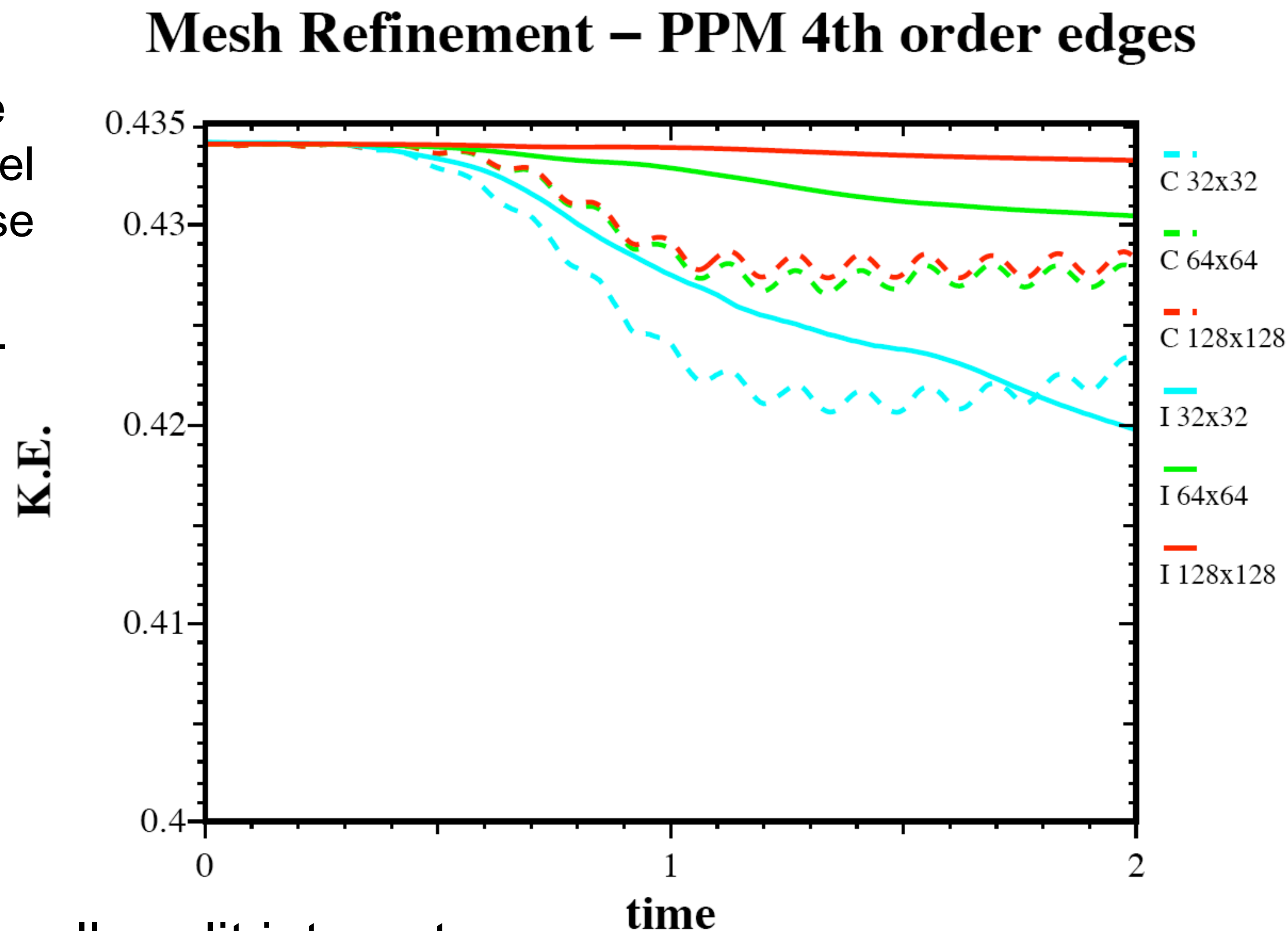
This is a convergent compressible sequence, converging at rate of 0.972, but it is more diffuse at 128x128 than the incompressible 32x32 calculation!
Converges to the "same" finite dissipation result as R-K.

Note that all the compressible curves turn up at late time and The 128x128 is more dissipative than the 64x64

*A dimensionally split integrator

The PPM* method performs a bit better than PLM, but its not sufficient.

When I refine one more level problems arise with the convergence.



*A dimensionally split integrator

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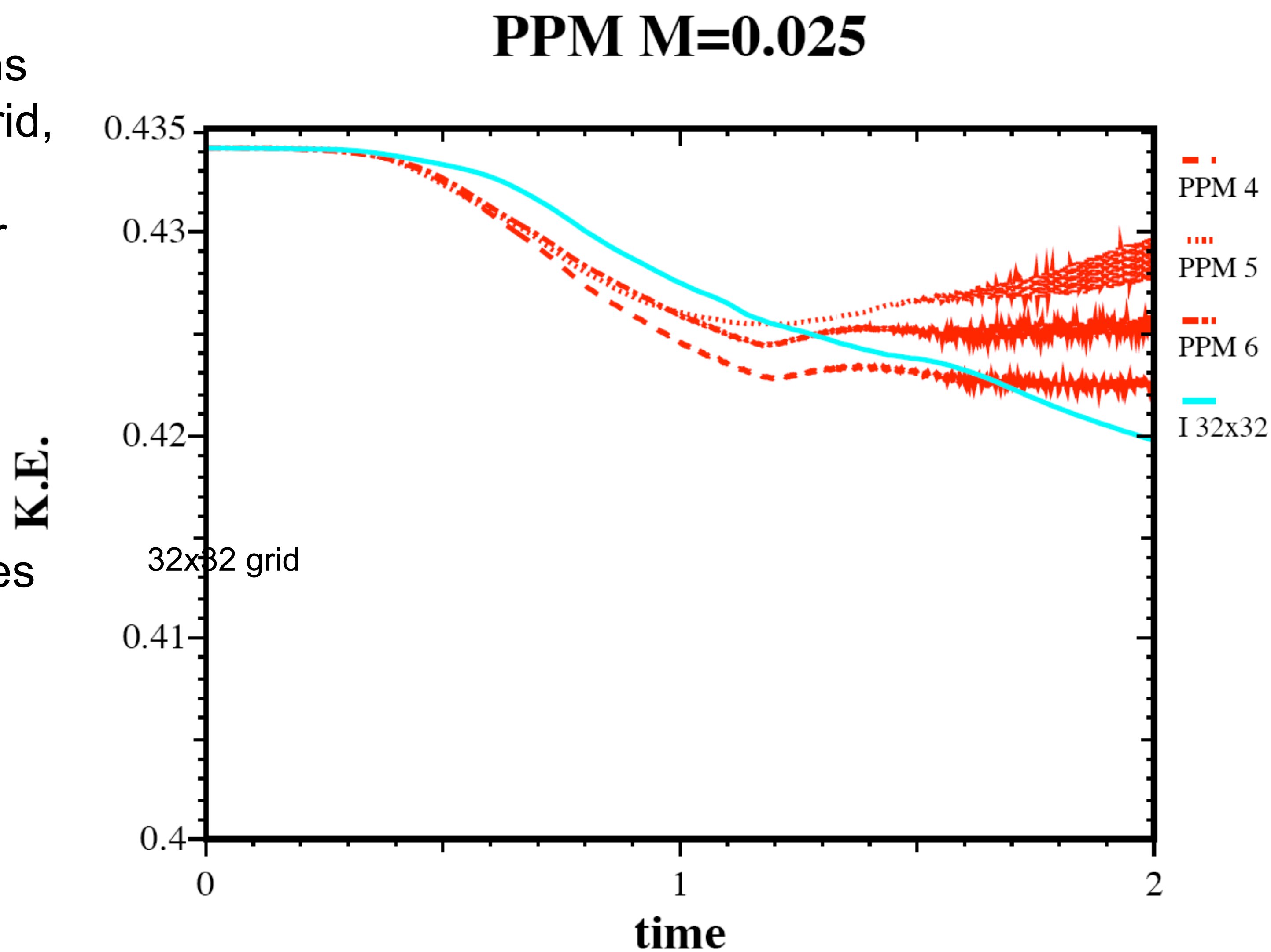
Using different edge value differencing, shows some interesting sensitivity.

All calculations on a 32x32 grid, and the structure near the end each calculation is related to the presence of nonlinear acoustic waves (shocklets).



Yet another advantage of PPM: Asymptotically preserving solutions

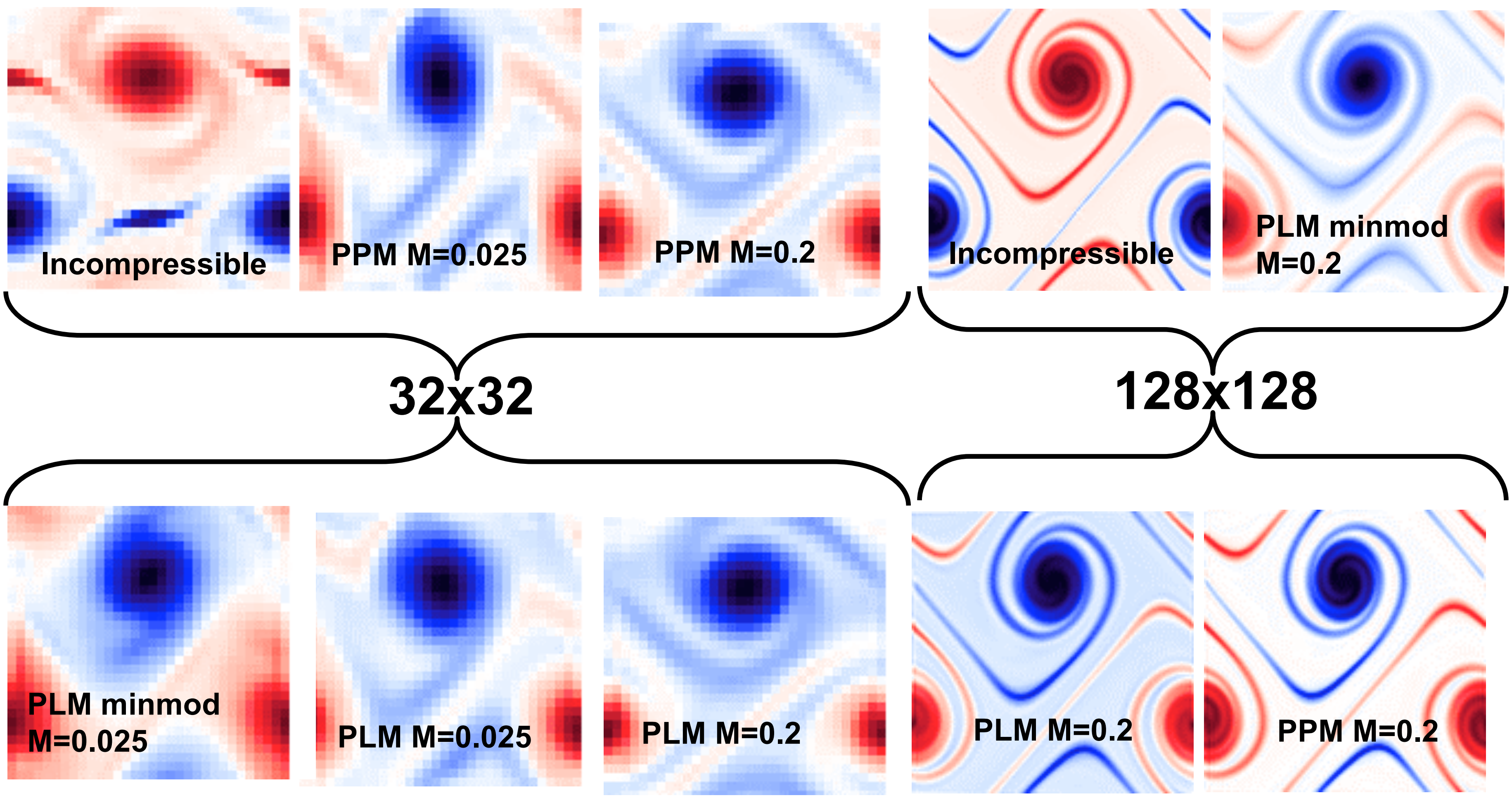
- **If one looks at solutions where the is an asymptotic structure, the truncation error can inhibit convergence, unless the small scale structure is resolved. PLM does this!**
  - **PPM: Continuous edge values as** $\Delta \mathbf{t} \rightarrow 0$
- **Example - Reaction system with a diffusive limit**

$$\partial_t \mathbf{u} + \partial_x \mathbf{v} = 0; \partial_t \mathbf{v} + \frac{1}{\varepsilon^2} \partial_x \mathbf{u} = -\frac{1}{\varepsilon^2} \mathbf{v} \Rightarrow \partial_t \mathbf{u}^{(0)} - \partial_x^2 \mathbf{u}^{(0)} = 0$$

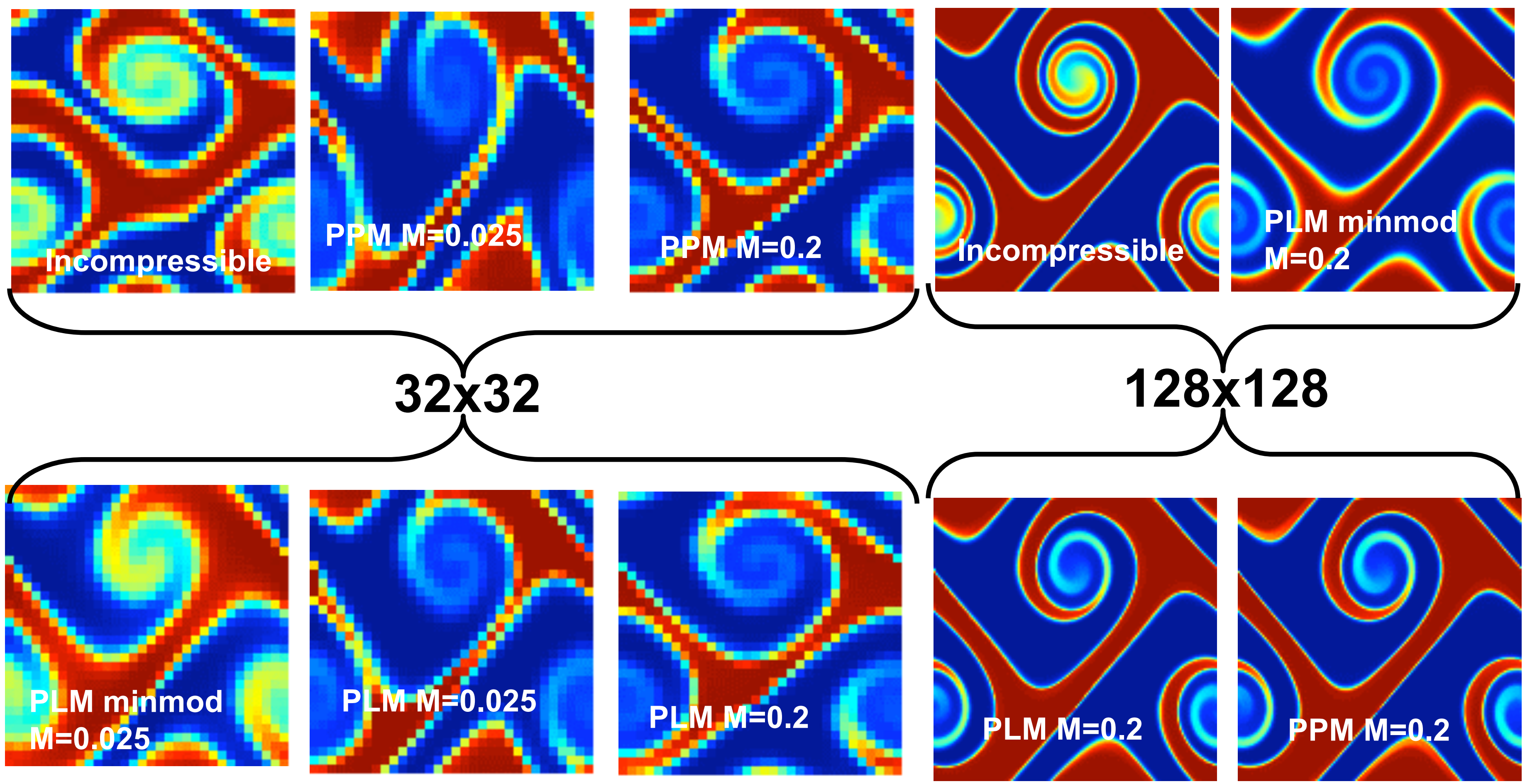
- **Example 2 - Acoustics in the zero(low)-Mach limit**

$$\partial_t \mathbf{u} + \partial_x \mathbf{v} = 0; \partial_t \mathbf{v} + \frac{1}{\varepsilon^2} \partial_x \mathbf{u} = \mathbf{0}; \lambda = \pm \frac{1}{\varepsilon}$$

Vorticity shows that determining the "best" solution is unclear.



Material motion results show that the "better" schemes to be worse!

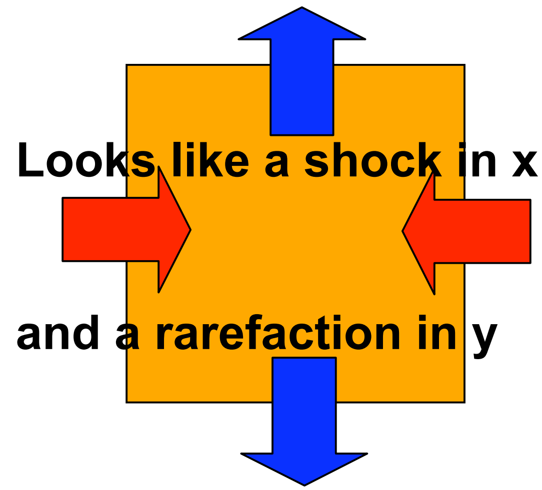


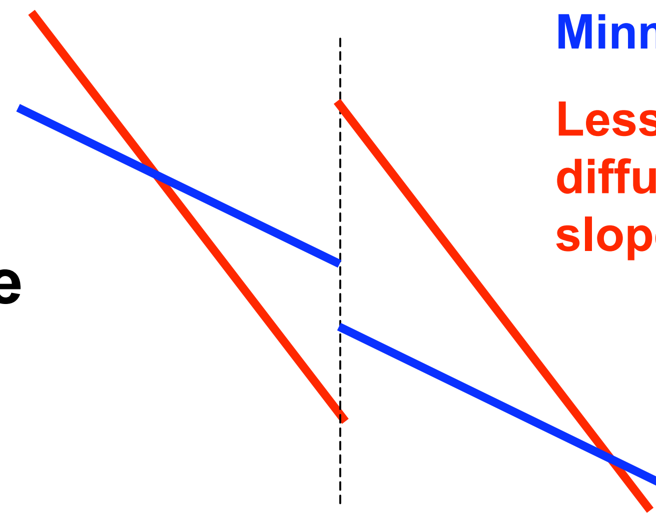
From these results we can draw some preliminary conclusions.

- PPM generally is less diffusive (of K.E.) than PLM, and centered edge-based approximations are (almost) physically and diffusively acceptable.
- The minmod PLM method is convergent, but is very diffusive, but appears better for material advection.
  - Results are relatively insensitive to Mach number.
- Less diffusive PLM methods behave unphysically late in time (t>1).
- The R-K (MOL) integrator with centered edge values is mesh convergent, but too diffusive as the Mach number descreases.
  - Upwind edges behave unphysically.

The methods misbehave because div(u)=0 looks like a shock.

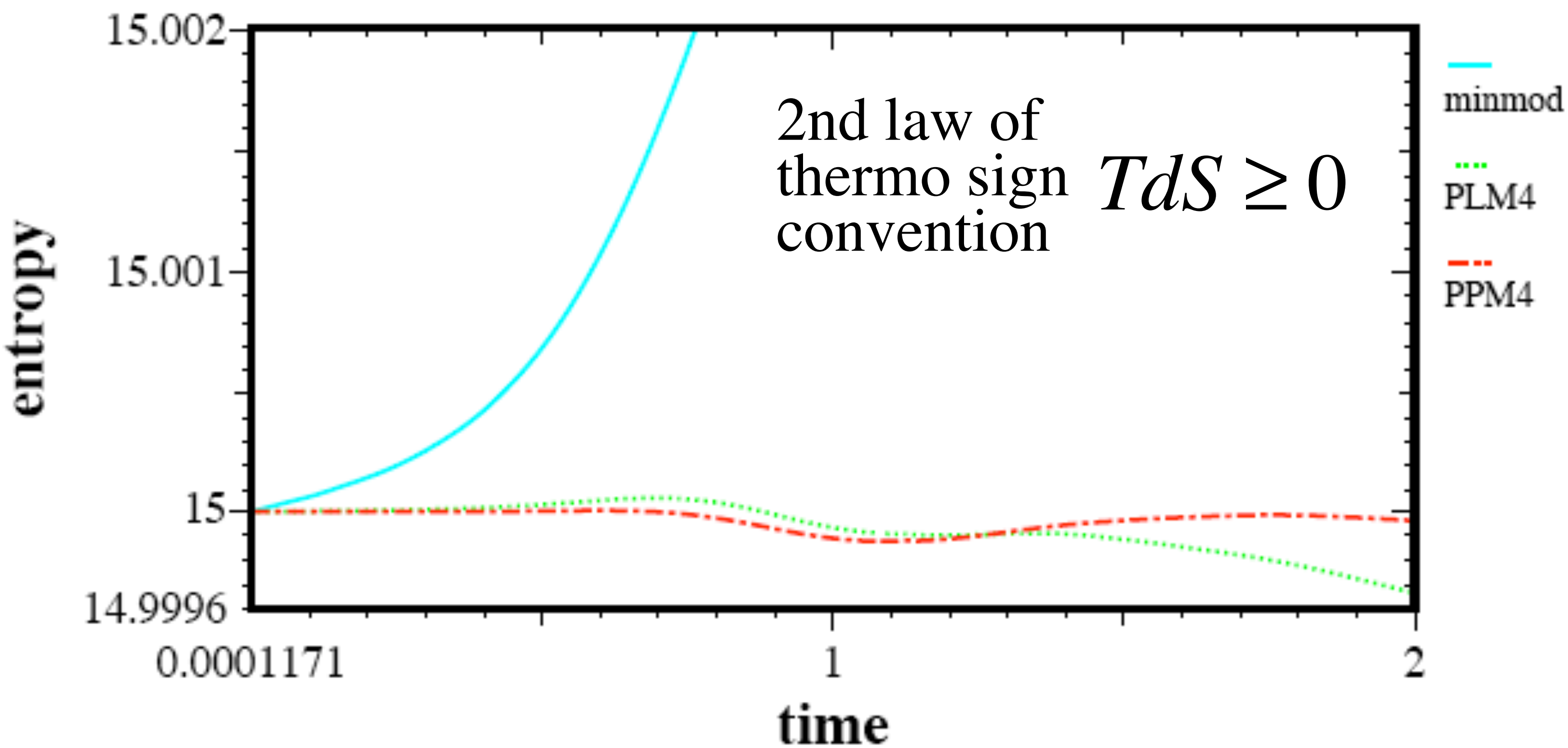
- **Div(u)=0 produces semi-permanent shock/rarefaction pairs cell-by-cell**
  - **Is shock dissipation appropriate?**
- **Schemes can produce meta-stable states that produce anti-diffusion without a shock then “healing” the unphysical state.**
  - **Small CFL numbers make the problem worse!**





Examining the evolution of entropy yields some disturbing results.

Any quality assessment that counts the “swirls” would Favor such violations since it will make the flow swirlier



Violations of the 2nd law of thermodynamics are more serious than other problems.

- The violations of the 2nd law of thermodynamics is more critical than too much dissipation.
- This implies that results are unphysical rather than simply inaccurate.
- The reason for this violation seem to be directly linked to the spatial differencing.
- The problem with violations of an “entropy condition” slowly gets worse as the Mach number decreases,
    - Points to a problem with the conditioning of the problem.

Why are there violation of the 2nd law?

- The problem is clearly associated with the “entropy” wave (but not shear!) carrying the density & energy changes.
- Effectively the low Mach number flow is poorly conditioned,

$$\frac{\partial}{\partial t}\begin{pmatrix}\rho\\ u\\ v\\ p\end{pmatrix}+\begin{pmatrix}u & \rho & 0 & 0\\ 0 & u & 0 & 1/\rho\\ 0 & 0 & u & 0\\ 0 & \rho c^2 & 0 & u\end{pmatrix}\frac{\partial}{\partial x}\begin{pmatrix}\rho\\ u\\ v\\ p\end{pmatrix}+\begin{pmatrix}v & 0 & \rho & 0\\ 0 & v & 0 & 0\\ 0 & 0 & v & 1/\rho\\ 0 & 0 & \rho c^2 & v\end{pmatrix}\frac{\partial}{\partial y}\begin{pmatrix}\rho\\ u\\ v\\ p\end{pmatrix}=0$$

- The condition number is the ratio of largest to smallest eigenvalues, $\lambda = \begin{pmatrix} u-c & u & u & u+c \end{pmatrix}$
  - The condition number can become infinite if *u* becomes small (as the Mach number decreases).

If incompressible flows are ill-conditioned what can be done?

- Past efforts have focused on preconditioning the system, thus basically removing some of the compressible character of the solutions in order to produce incompressible solutions (not practical).
- Another common approach is when problems are found dissipate them!
- The minmod limiter does this naturally, but its overkill, it dissipates the entire flow.
- The trick is to detect the problems and deal with them locally where the problem occurs.
  - Careful examination found that all the problems occur in the entropy modes in compressible flow,

$$\alpha_S = \frac{\Delta\rho}{\rho} - \frac{\Delta p}{\rho c^2}$$

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Is incompressibility entirely physical?

- No, its not. Certainly not *inviscid* incompressibility.
- For starters incompressibility has <u>infinite signal speeds</u>, fluids do not, finite speed of propagation is necessary.
  - There is no second law, and vanishing viscosity is not generally an applied principle..
  - …except for the derivation of Margolin, Rider & Grinstein JOT 2006. The finite scale equations have solutions based on vanishing viscosity.
- Incompressibility does not have known mechanisms for producing singularities, but they are necessary to explain the <u>fundamental behavior of turbulence</u>.
- Compressible flows produce singularities, i.e. shocks under almost any conditions.

The production of dissipation without viscosity is essential for many processes.

- **For shock waves in the limit of weak shocks**

$$T\,\Delta S = -\frac{\mathcal{G}}{6c}\left(\Delta u\right)^3;\frac{\Delta u}{c} \to 0;\mathcal{G} = -\frac{\gamma V^2}{2p}\frac{\partial^2 p}{\partial V^2}$$

- **For three dimensional turbulence**

$$\frac{4}{5}\left\langle \frac{\partial K}{\partial t}\right\rangle = \left\langle \frac{\left(\Delta_\ell u\right)^3}{\ell}\right\rangle$$

$$\frac{\Delta K}{\Delta t} = \frac{\mathcal{G}}{6}\frac{\left(\Delta u\right)^3}{\ell} \to \Delta t \approx \frac{\ell}{c}$$

- **For Burgers' equation, comes from the jump conditions**

$$\frac{1}{12}\left\langle \frac{\partial K}{\partial t}\right\rangle = \left\langle \frac{\left(\Delta_\ell u\right)^3}{\ell}\right\rangle$$

- **The weak shock limit and the zero Mach number limit are one and the same! What is the difference?**

$$\boxed{\Delta u/c \to 0}$$

The difference between the low Mach number limit and weak shock is subtle.

Compressible flows can shock at any Mach number!

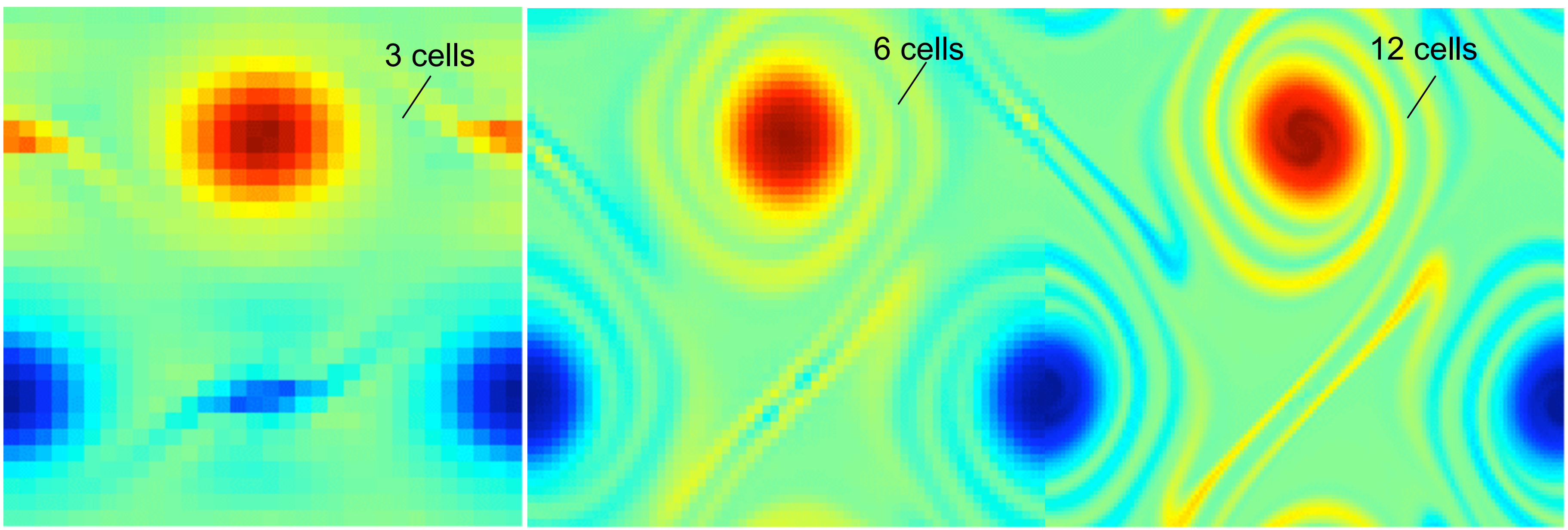
- The difference comes down to the implied smoothness of the flow.
- A weak shock limit has an implied discontinuity, a shock (differs from adiabatic at 3rd order),
- …while the incompressible flow is a well-conditioned nice flow.
- What really happens? Under what conditions do physical (inviscid) flows fail to shock?

$$\left(\frac{\partial u}{\partial x}\right)^{\tau} = 1 \Big/ \left[1 + \tau \mathcal{G}\left(\frac{\partial u}{\partial x}\right)^{0}\right]$$

  - Do incompressible flows deplete the nonlinear mechanism for shock formation?

Evidence shows that incompressible flows steepen and produce shock-like structures.

- **By shock-like this means that the flow structures achieve a thickness that is linearly dependent on resolution (3 zones wide regardless of mesh density)**
- **This occurs quite clearly in the shear layer problem (and many others!)**



Summary

- Low-Mach number shear flows are a distinct challenge for compressible solvers
- Various methods perform well with issues for low-Mach shear (more diffusive limiters have advantages)
  - Dimensionally split solvers are not too diffusive, but can show 2nd law of thermo violations
  - MOL+centered differencing is convergent in mesh refinement, but not Mach number
- These errors can be viewed as arising from the ill-conditioning of the system of equations.
- Incompressibility as an appropriate model should be examined critically.